INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 5, 305–310 (1985)

THE CELL REYNOLDS NUMBER MYTH

H. DOYLE THOMPSON*, BRENT W. WEBB† AND JOE D. HOFFMAN*

School of Mechanical Engineering, Purdue University, West Lafayette, Indiana 47907, U.S.A.

SUMMARY

The commonly accepted linear stability analysis for the forward-time centred-space (FTCS) algorithm applied to the transport equation has led to the concept of a cell Reynolds number restriction on the spatial grid size. This paper shows where the commonly accepted original analysis is in error and presents the correct stability restrictions, which are restrictions on the time step, not the spatial grid size. There is no cell Reynolds number restriction. The results are confirmed by numerical computations for the two-dimensional driven cavity problem.

INTRODUCTION

The study of the numerical stability of finite difference algorithms usually starts with the analysis of the forward-time centred-space (FTCS) algorithm applied to a model equation such as the one-dimensional transport equation

$$f_t = -uf_x + \alpha f_{xx} \tag{1}$$

Such a didactic study provides insights into some of the sources of numerical instability, illustrates the relationship between the convective term uf_x and the diffusion term αf_{xx}, and can be used to demonstrate such concepts as artificial viscosity and numerical diffusion. Furthermore, in spite of its limitations, the FTCS algorithm is still used for many practical calculations. Therefore, owing to its educational, historical and practical value, it is important that the stability analysis for the FTCS algorithm be correct and accurate.

Unfortunately, there remains in the literature and among practitioners the idea that there is a cell Reynolds number restriction that must be satisfied for the FTCS algorithm to be numerically stable. That myth is almost as hard to dispel as the flat earth theory in Columbus's time or the idea that the sun revolved around the earth in the days of Galileo.

The concept of a cell Reynolds number restriction evolved from the 1964 work of Fromm.[1] He used the now classical von Neumann stability analysis technique[2] to determine necessary stability limits on the two-dimensional vorticity transport equation, and mistakenly arrived at the conclusion that

$$0 \leqslant C \leqslant 2d \leqslant 1 \tag{2}$$

where C is the Courant number and d is the diffusion number, defined by

$$C = u\Delta t/\Delta x \quad \text{and} \quad d = \alpha\Delta t/\Delta x^2 \tag{3}$$

*Professor
†Research Assistant

0271–2091/85/04 0305–06$01.00

Received 17 January 1984
Revised 12 June 1984

Equation (2) corresponds to the well-known criteria

$$\Delta t \leqslant \Delta x^2/2\alpha \tag{4}$$

and

$$R_c \equiv u\Delta x/\alpha \leqslant 2 \tag{5}$$

where R_c is the so called cell Reynolds number. Equations (2)–(5) are actually the one-dimensional equivalent of Fromm's two-dimensional result. Their derivation contains an important error, but in the two-dimensional setting there is also a second (distinct) error in Fromm's analysis. That second error is corrected by the recent work of Hindmarsh, Gresho and Griffiths[3] who appear to be the first to have derived the correct stability limits for the two- and three-dimensional cases. To illustrate the error that led to the cell Reynolds number concept, it is sufficient to consider only the one-dimensional case.

Equation (4) is a correct result but, as is shown below, equation (5) is both overly restrictive and very misleading since it requires a spatial grid size limitation for stability where in reality none exists. As early as 1968 Hirt[4] arrived at the correct criteria for the one-dimensional case using an entirely different approach to the stability analysis. That result received little notice, however, and the cell Reynolds number myth has propagated and amplified throughout the numerical analysis community. Examples in the literature include the works of Marshall,[5] Noye,[6] Olson and Tuann[7] and Torrance *et al.*[8] The result also appears in educational texts and reference books, e.g. those of Roache,[9] Mitchell and Griffiths[10] and Lapidus and Pinder.[11]

Many practitioners know that the cell Reynolds number limit can be violated (excessively) and still obtain stable numerical results, and many have suspected that something is not quite right. Roache[9] even went so far as to show that the result obtained by considering the geometric aspects of the stability ellipse obtained from the von Neumann analysis gives a result that is not consistent with the cell Reynolds number restriction, but he did not pursue the discrepancy.

In addition, the misconception has led to some unusual schemes that misleadingly appear to have merit. For example Chien[12] developed an elaborate scaling procedure for the FTCS algorithm which he claimed had the main advantage of eliminating the stability limit on the grid size. Careful analysis shows, however, that Chien's scaling does not change the stability limits of the algorithm, and that the apparent improvement, shown by numerical experiment, is simply a manifestation of the fact that the assumed cell Reynolds number limit never really existed.

Rigal[13] clearly states that there is not a cell Reynolds number stability limit, but unfortunately his two-dimensional limit, although necessary, is not sufficient for stability. Leonard[14] appears to have been the first to publish the correct one-dimensional result. Clancy[15] independently obtained the correct one-dimensional result. More recently, Hindmarsh, Gresho and Griffiths[3] have more thoroughly analysed the stability of the general multidimensional case. Whereas Leonard,[14] Clancy[15] and Hindmarsh *et al.*[3] clearly state that Fromm's original work is in error, they each resort to other means (e.g. the stability ellipse approach discussed by Roache[9]) to obtain the correct limits. It is believed that the following derivation is the first which shows where Fromm went wrong and brings all the analysis methods that depend on the von Neumann approach into agreement.

CORRECTING FROMM'S ANALYSIS

The von Neumann stability analysis is based on the representation of numerical errors at a specific time level by an infinite Fourier series, and the examination of the relative magnitudes of the various Fourier components of the errors at successive time steps. The complex amplification factor, G, for the FTCS algorithm applied to equation (1) is

$$G = 1 + 2d(\cos\theta - 1) - \mathrm{i}C\sin\theta \tag{6}$$

where $i = \sqrt{(-1)}$ and θ is a parameter that can take on any value between $\pm\infty$. The von Neumann analysis requires that for numerical stability the magnitude of G be no greater than unity for any value of θ. Multiplying G by its complex conjugate $\bar{G}$, the stability requirement can be expressed as

$$G\bar{G} = [1 + 2d(\cos\theta - 1)]^2 + C^2(1 - \cos^2\theta) \leqslant 1 \tag{7}$$

The stability criteria (i.e. restrictions on the permissible values of d and C) result from examination of equation (7) for all values of θ. This can be done by several methods, but each must produce the same result. Fromm[1] chose to infer the stability criteria from examination of the critical points (maxima and minima) of the function $G\bar{G}$. In that examination he also chose $\cos\theta$ as the independent variable. Thus, substituting x for $\cos\theta$, the problem becomes one of finding the values of C and d for which

$$G\bar{G} = [1 + 2d(x - 1)]^2 + C^2(1 - x^2) \leqslant 1 \tag{8}$$

in the interval $-1 \leqslant x \leqslant 1$.

At $x = 1$, $G\bar{G} = 1$, and no limitation is present. At $x = -1$, $G\bar{G} = (1 - 4d)^2$. Thus, one obvious limitation is $0 \leqslant d \leqslant 1/2$ which leads to the time step limitation given by equation (4). A second limitation can be deduced from the behaviour of the function $G\bar{G}$. As illustrated in Figure 1, there are three different types of behaviour for the function $G\bar{G}$ in the range $-1 \leqslant x \leqslant 1$. They are that $G\bar{G}$ exhibits a maximum in the interval (curve (a)), a minimum in the interval (curve (b)), or neither a maximum nor minimum in the interval (curve (c)). Case 1 (curve (a)) represents unstable solutions, since the maximum value would be greater than either end point, and thus exceed unity. Case 2 (curve (b)) represents a set of stable solutions provided that $0 \leqslant d \leqslant 1/2$ and case 3 (curve (c)) represents another set of stable solutions for $0 \leqslant d \leqslant 1/2$.

Using well-known methods to solve for the maximum of the function $G\bar{G}$, the results in Table I are obtained.

The result for case 1 was obtained by requiring that $G\bar{G}$ have a maximum in the interval $-1 \leqslant x \leqslant 1$. Case 2 represents situations for which a minimum exists. Case 3 represents situations

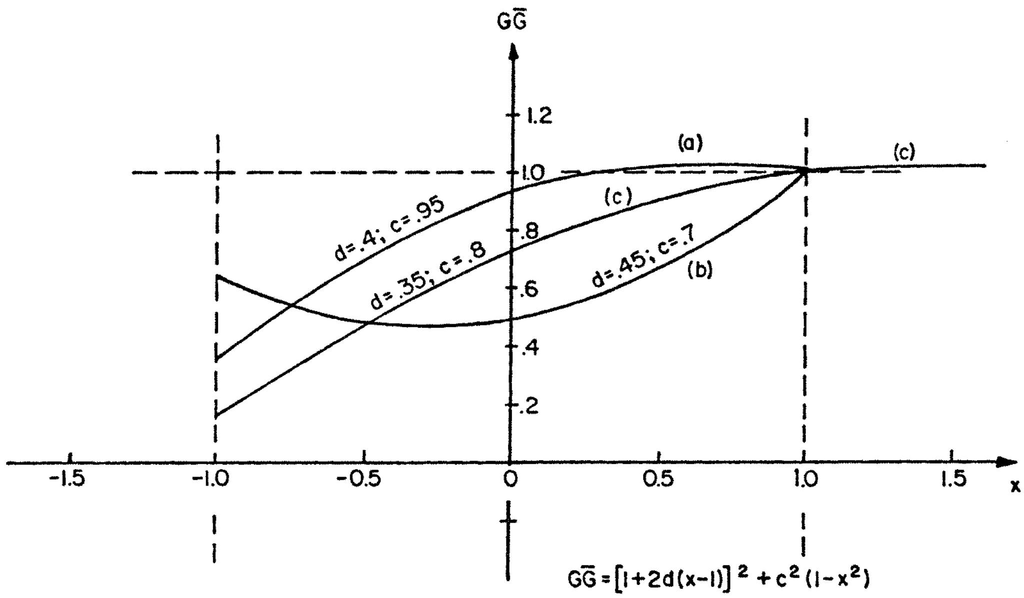

Figure 1. $G\bar{G}$ as a function of x for various combinations of C and d

Table I

Case	Curve	Criteria	Comments
1	(a)	$C^2 > 2d$	unstable
2	(b)	$0 \leqslant C \leqslant 2d \leqslant 1$	stable
3	(c)	$0 \leqslant C^2 \leqslant 2d \leqslant 1$	stable

for which a maximum occurs at $x > 1$. Since there are no values of θ for which $x = \cos\theta > 1$, the values of $G\bar{G}$ at $x > 1$ are of no interest. It is an unreal set of solutions that was introduced by changing the independent variable from θ to $\cos\theta$.

Fromm's error was failing to recognize the set of stable solutions represented by case 3. He thereby obtained overly restrictive and misleading conditions given by equation (2).

The correct condition for stability is the union of stable solutions represented by cases 2 and 3, namely

$$0 \leqslant C^2 \leqslant 2d \leqslant 1 \tag{9}$$

which reduces to equation (4) plus the restriction $R_c C \leqslant 2$ or

$$\Delta t \leqslant 2\alpha/u^2 \tag{10}$$

Equation (10) is a restriction on the time step and is independent of the time step restriction given by equation (4). Notice that neither equation (4) nor equation (10) contains a restriction on the spatial grid size.

Equations (4) and (5) are sufficient for stability, since they are more conservative than the sufficient and necessary conditions, equations (4) and (10). However, the adoption of equations (4) and (5) as necessary conditions leads to incorrect conclusions such as the cell Reynolds number criterion and the commonly held belief that it is not possible to compute stable solutions at high Reynolds numbers using the FTCS algorithm without a small spatial grid.

The mesh refinements necessitated by equation (5) are definitely more costly in terms of computer storage and computation time than the reduction in time step prescribed by equation (10). Those who have stabilized the algorithm at higher Reynolds number by mesh refinements have done so because the smaller grid size also requires a smaller time step due to the limitation in equation (4). The conclusive stabilizing factor is the reduction in time step, not the refinement of the spatial mesh.

It should be noted that this conclusion deals only with the stability aspects and not the accuracy of the solution. Spatial grid refinements at higher Reynolds number will undoubtedly produce a more accurate solution.

TWO-DIMENSIONAL CRITERIA

The general two-dimensional transport equation, analogous to equation (1), is

$$f_t = -uf_x - vf_y + \alpha_1 f_{xx} + \alpha_2 f_{yy} \tag{11}$$

The stability criterion from the von Neumann analysis (comparable to equation (7)) is

$$G\bar{G} = [1 + 2d_1(\cos\theta_1 - 1) + 2d_2(\cos\theta_2 - 1)]^2 + [C_1 \sin\theta_1 + C_2 \sin\theta_2]^2 \leqslant 1 \tag{12}$$

for all values of θ_1 and θ_2, where

$$C_1 = \frac{u\Delta t}{\Delta x}; \quad C_2 = \frac{v\Delta t}{\Delta y}; \quad d_1 = \frac{\alpha_1 \Delta t}{\Delta x^2}; \quad d_2 = \frac{\alpha_2 \Delta t}{\Delta y^2} \tag{13}$$

Fromm[1] simplified his original analysis by assuming that $\theta_1 = \theta_2$. He did note, however, that the criteria so obtained can only be less stringent than the true criteria, that is, that simplification is under-restrictive and leads to a necessary condition that is not sufficient. That logical error has been masked heretofore by the conservative cell Reynolds number myth.

For the special case of $\theta_1 = \theta_2$, equation (12) can be written

$$G\bar{G} = [1 + 2(d_1 + d_2)(\cos\theta - 1)]^2 + (C_1 + C_2)^2(1 - \cos^2\theta) \leqslant 1 \tag{14}$$

for all values of θ. By analogy to equation (7), the resulting necessary stability criteria are

$$0 \leqslant (C_1 + C_2)^2 \leqslant 2(d_1 + d_2) \leqslant 1 \tag{15}$$

Hindmarsh, Gresho and Griffiths[3] have shown that the special condition $\theta_1 = \theta_2$ leads to a non-sufficient restriction (i.e. equation (15) is necessary but not sufficient), and have determined the correct necessary and sufficient conditions by treating θ_1 and θ_2 as independent variables. Their result is generalized for n-dimensions. The correct result for the two-dimensional case is

$$0 \leqslant 2(d_1 + d_2) \leqslant 1 \tag{16}$$

and

$$0 \leqslant C_1^2/2d_1 + C_2^2/2d_2 \leqslant 1 \tag{17}$$

In terms of a time-step limitation, equations (16) and (17) give

$$\Delta t \leqslant 1 \bigg/ \left[\frac{2\alpha_1}{\Delta x^2} + \frac{2\alpha_2}{\Delta y^2}\right] \tag{18}$$

and

$$\Delta t \leqslant 1 \bigg/ \left[\frac{u^2}{2\alpha_1} + \frac{v^2}{2\alpha_2}\right] \tag{19}$$

There is no limitation on the spatial grid.

The stability criteria given by equations (18) and (19) were tested using numerical experimentation for the FTCS algorithm applied to the two-dimensional driven cavity problem using the stream-function–vorticity approach. Because of the non-linearity of the problem (u and v are not constant) the demarcation between stable and unstable calculations is not as sharp as for linear problems. None the less, the result was clear in that stable solutions (not necessarily accurate solutions) are always obtained when equations (18) and (19) are locally satisfied. Furthermore, unstable results are obtained when the criteria of equation (15) are satisfied but equation (17) is violated.

Finally, the numerical results show that the local cell Reynolds number limitation given by Fromm can be violated by nearly two orders of magnitude while obtaining well-behaved, stable, reasonable solutions. Spatial grid refinements were required for enhanced accuracy, but offered no advantage with respect to the stability of the algorithm.

CONCLUSIONS

The necessary and sufficient conditions for stability of the FTCS algorithm have finally been corrected after 20 years. The correct criteria are limitations on the admissible time step and do not contain a space grid limitation. The idea of a cell Reynolds number restriction is a myth.

REFERENCES

1. J. Fromm, 'The time-dependent flow of an incompressible viscous fluid', in B. Alder, S. Fernbach and M. Rotenberg (eds), *Methods in Computational Physics*, **3**, Academic Press, New York, 1964.

2. G. G. O'Brien, M. A. Hyman and S. Kaplan, 'A study of numerical solutions of partial-differential equations', *J. Math. and Phys*, **29**, 223–251 (1950).
3. A. C. Hindmarsh, P. M. Gresho and D. F. Griffiths, 'The stability of explicit Euler time-integration for certain finite difference approximations of the multi-dimensional advection–diffusion equation', *Int. j. numer. methods fluids*, **4**, 853–897 (1984).
4. C. W. Hirt, 'Heuristic stability theory for finite–difference equations', *J. Comp. Phys.* **2**, 339–355 (1968).
5. G. Marshall, 'Computational aspects of a viscous incompressible fluid', in R. D. Richtmeyer (ed.) *Proceedings of the 4th International Conference on Numerical Methods in Fluid Dynamics*, University of Colorado, in *Lecture Notes in Physics*, **35**, Springer-Verlag, New York, 1974, pp. 269–273.
6. B. J. Noye, 'An introduction to finite difference techniques', in B. J. Noye (ed.) *Proceedings of an International Conference on the Numerical Simulation of Fluid Dynamic Systems*, Monash University, Melbourne, Australia, North Holland-Elsevier, New York, 1976, pp. 56–85.
7. M. D. Olson and S. Y. Tuann, 'Computing methods for recirculating flow in a cavity', in C. Taylor, K. Morgan and C. A. Brebbia (eds) *Proceedings of the 1st International Conference, Numerical Methods in Laminar and Turbulent Flow*, University College of Swansea, United Kingdom, Wiley, New York, 1978, pp. 873–883.
8. K. Torrance, R. Davis, K. Eike, P. Gill, D. Gutman, A. Hsui, S. Lyons and H. Zien, 'Cavity flows driven by buoyancy and shear', *J. Fluid Mech.*, **51**, 221–246 (1972).
9. P. J. Roache, *Computational Fluid Dynamics*, Hermosa Publishers, Albuquerque, New Mexico, 1972.
10. A. R. Mitchell and D. F. Griffiths, *The Finite Difference Method in Partial Differential Equations*, Wiley, New York, 1980.
11. L. Lapidus and G. Pinder, *Numerical Solutions to Partial Differential Equations*, Wiley, New York 1982.
12. J. C. Chien, 'A general finite-difference formulation with application to Navier–Stokes equations', *J. Comp. Phys.*, **20**, 268–278 (1976).
13. A. Rigal, 'Stability analysis of explicit finite difference schemes for the Navier–Stokes equations', *Int. j. num. meth. eng.*, **14**, 617–620 (1979).
14. B. P. Leonard, 'Note on the von Neumann stability of the explicit FTCS convective diffusion equation', *Appl. Math. Modeling*, **4**, 401–402 (1980).
15. R. M. Clancy, 'A note on finite differencing of the advection–diffusion equation', *Mon. Wea. Rev.*, **109**, 1807–1809 (1981).